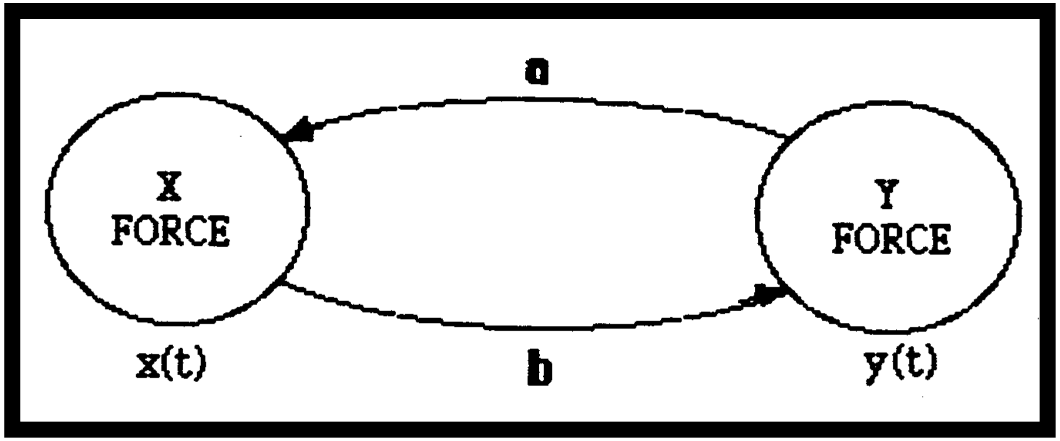
Introduction to Lanchester Equations

This section introduces perhaps the most important model of combat, Lanchester's equations

Dr. Frederick W. Lanchester



$$\frac{dx}{dt} = -ay \quad and \quad \frac{dy}{dt} = -bx$$

Motivation/Background

- Many important models use derivatives/extensions of Lanchester's work
  - For example: VIC, CEM, THUNDER, JWARS, ITEM
- This section illustrates how simple models can produce insightful information
  - Principle of concentration of forces in modern warfare
- This model (i.e., Lanchester equations) will be used to introduce many important concepts that we will use throughout the course

Section Learning Objectives

- Understand Lanchester's original two models
  - *Square law* and *linear law*
- Be able to determine who wins, how long the battle lasts, the number of survivors on each side, and give battle traces
- Learn about enrichments and extensions to Lanchester's homogenous models
  - Ambush model
- Be able to numerically implement and exercise Lanchester equations
- Extend what we've learned to heterogeneous Lanchester models
- Introduction to where the attrition coefficients come from

Outline

- Lanchester's initial models
  - Square law and linear law
- Some calculations (victory conditions, battle trace)
- Some extensions to Lanchester models
- Enriching Lanchester
- Solving Lanchester numerically
- Heterogeneous Lanchester
- A few words on getting the attrition coefficients

Lanchester's Pioneering Work

- Published in 1914
- Lanchester was a prominent British inventor
  - Built first British automobile
- Models were used to argue the benefits of force concentration in modern warfare
  - Lanchester was interested in applying the models to the burgeoning area of air combat
- Related efforts (sometimes see CLO equations)
  - Osipov (1915) studied historical battles
    - Of course, the Russians claim propriety
  - J.V. Chase (1902) developed similar Navy models
    - Prior Navy work not declassified till 1972!

Lanchester's Two Models

- Lanchester's (1914) differential equation attrition models
- Purely hypothetical study of "The principle of concentration of forces"
- Lanchester's two models
  - assuming **homogeneous forces**

$$\boxed{\frac{dx}{dt} = -ay \quad and \quad \frac{dy}{dt} = -bx}$$

- Modern Combat ==>

$$\boxed{\frac{dx}{dt} = -axy \quad \text{and} \quad \frac{dy}{dt} = -bxy}$$

- Ancient Combat ==>

Lanchester's Model of "Modern Warfare"

- Lanchester's (1914) differential equation attrition models
- Two forces, *X* and *Y*
- Idea: "The rate at which a force is depleted is proportional to the size of the enemy and the individual capability of the enemy."

$$\boxed{\frac{dx}{dt} = -ay(t) \quad \text{and} \quad \frac{dy}{dt} = -bx(t)}$$

Lanchester's Model of "Modern Warfare"
Definitions

$$\boxed{\frac{dx}{dt} = -ay \quad \text{and} \quad \frac{dy}{dt} = -bx}$$

- *Definitions:*
  - t = time
  - $x = x(t) \geq 0$: x force level at time t
  - $y = y(t) \geq 0$: y force level at time t
  - a and b are "attrition coefficients" with units (for a) of

$$\boxed{\frac{x \text{ casualties}}{(\text{time}) \times (\text{one y participant})}}$$

- One simple interpretation:
  a = (firing rate of <u>one</u> y) $\times$ Pr(a shot from y kills an x)
- Think: a = rate at which one y kills x forces
- In practice, a lot goes into determining a and b

Modern Warfare Continued

- $x_0 = x(0)$ = initial *X* force level
- $y_0 = y(0)$ = initial *Y* force level

– This model is also known as "square law," "aimed fire," or "conventional combat"

– What this model assumes/requires
  - homogeneous forces
  - constant attrition rate coefficients
  - can aim (mass fires) at targets
  - coordination of fires

The “Square Law” State Equation

- The *state equation* relates force levels over time

$$\boxed{b\left(x_o^2 - x(t)^2\right) = a\left(y_o^2 - y(t)^2\right)}$$

- Let’s Derive it

- Key is $\frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{dx}{dy} = \frac{ay}{bx}$

Lanchester's Model of Ancient Warfare

$$\frac{dx}{dt} = -axy \text{ and } \frac{dy}{dt} = -bxy$$

- Idea: each sides losses are proportional to both the other side's force level and their own force level
- *Ancient warfare's* coupled differential equations =>
  - where:
    - t = time
    - $x = x(t) = x$ force level at time t
    - $y = y(t) = y$ force level at time t
    - a and b are "attrition coefficients" with units (for a) of

$$\frac{\text{x casualties}}{(\text{time}) \times (\text{one x participant}) \times (\text{one y participant})}$$

    - Note: It doesn't make sense to compare attrition coefficients among different models (as they have different interpretation)

Lanchester's Model of Ancient Warfare
"Area Fire" Interpretation

$$\boxed{\frac{dx}{dt} = -axy \quad \text{and} \quad \frac{dy}{dt} = -bxy}$$

- One simple interpretation (for today's combat = area or artillery fire):
  a = (firing rate of *one y*) · (A_L/A_T)
  - where:
    - A_L = lethal area of one shot from y forces
    - X units are uniformly scattered points
    - A_T = area occupied by x forces ($A_L << A_T$)
    - see picture

Ancient Warfare Continued

- $x_0 = x(0)$ = initial x force level
- $y_0 = y(0)$ = initial y force level

– Sometimes called “area fire,” “linear law”

– What this requires
  - homogeneous forces
  - constant attrition rate coefficients
  - can not aim (mass fires) at targets--Lanchester thought of a series of one-on-one duels.

The "Linear Law" State Equation

- The state equation

$$\boxed{b\left(x_o - x(t)\right) = a\left(y_o - y(t)\right)}$$

- Let's Derive it…not or SBES (same idea as square law).

- key is $\dfrac{\frac{dx}{dt}}{\frac{dy}{dt}} = \dfrac{dx}{dy} = \dfrac{a}{b} = \text{a constant independent of x and y}$

Numerical example of linear law

- The state equation => $\boxed{b\left(x_o - x(t)\right) = a\left(y_o - y(t)\right)}$

  => $\boxed{y(t) = y_o - \frac{b}{a}\left(x_o - x(t)\right)}$
- Suppose *a* = *b* = .0001, x_0 = 100, and we fight-to-the-finish.

y_0	y_F	y_L
100	0	100
200	100	100
400	300	100
800	700	100

Numerical example of square law

- The state equation => $b\left(x_o^2 - x(t)^2\right) = a\left(y_o^2 - y(t)^2\right)$
  => $y(t) = \sqrt{y_o^2 - \frac{b}{a}\left(x_o^2 - x(t)^2\right)}$

- Suppose $a = b = .01$, $x_0 = 100$, and we fight-to-the-finish.

y_0	y_F	y_L
100	0	100
200	173.2	26.8
400	387.3	12.7
800	793.7	6.3

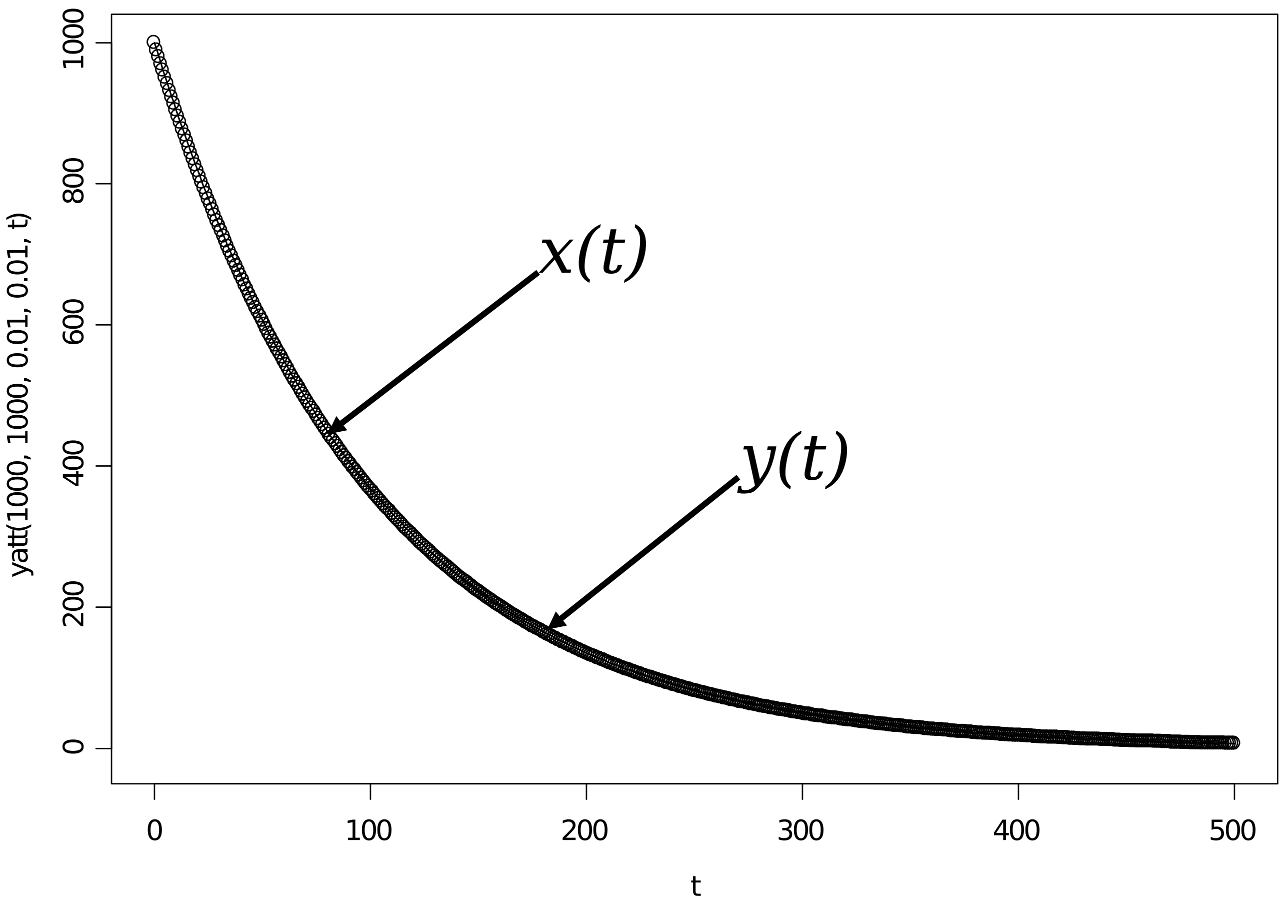
Lanchester Concluded

- In modern war there is added benefit to concentration of forces!
  - **it is an "important truth" that "the fighting strength of a force can be represented by the square of its numerical strength."**
- Mathematical explanation of
  - (a) **"Divide and conque**

  - (b) **"God is on the side of the big battalio**

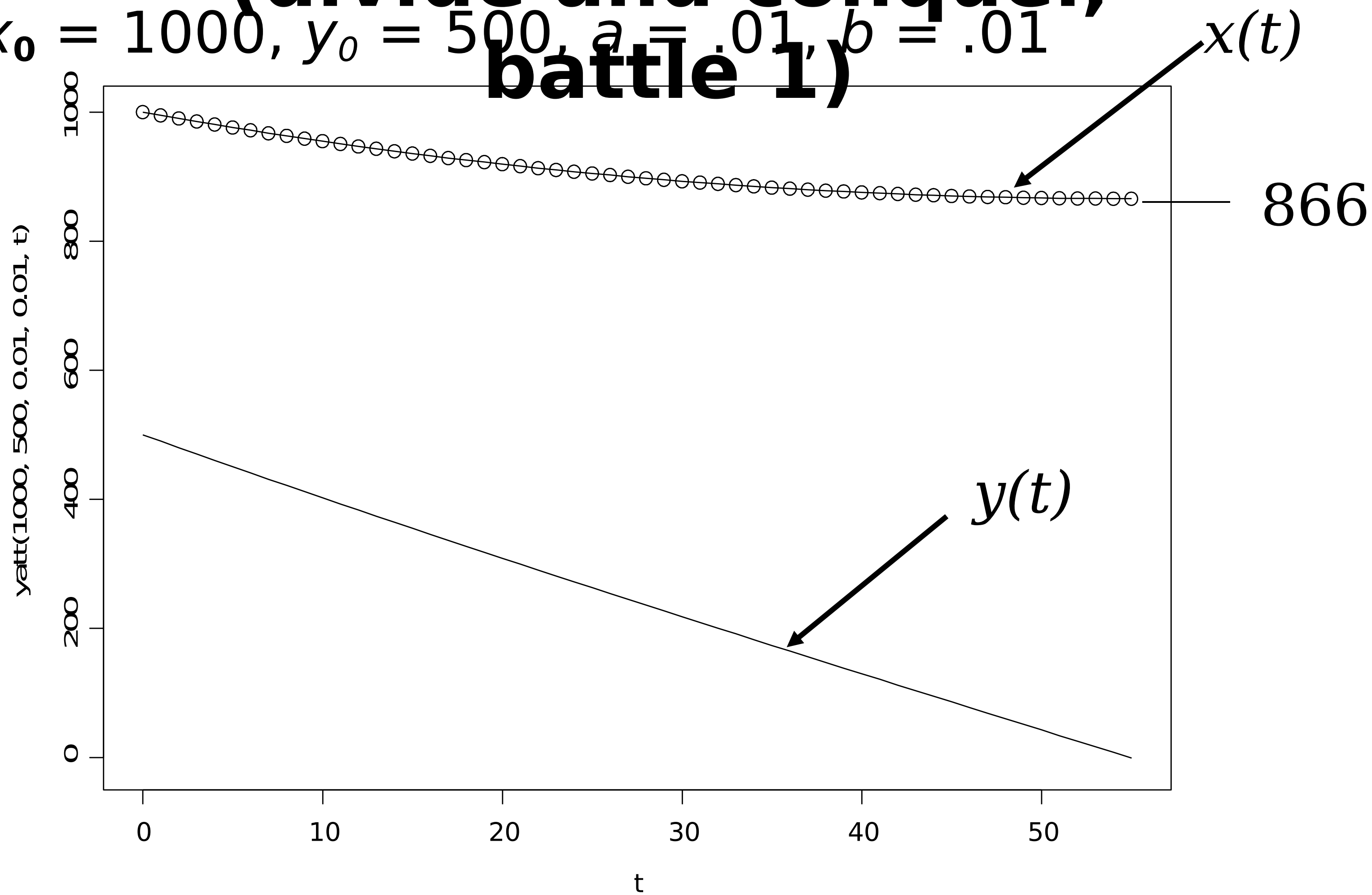
Example: Lanchester Square

- $x_0 = 1000$, $y_0 = 1000$, $a = .01$, $b = .01$



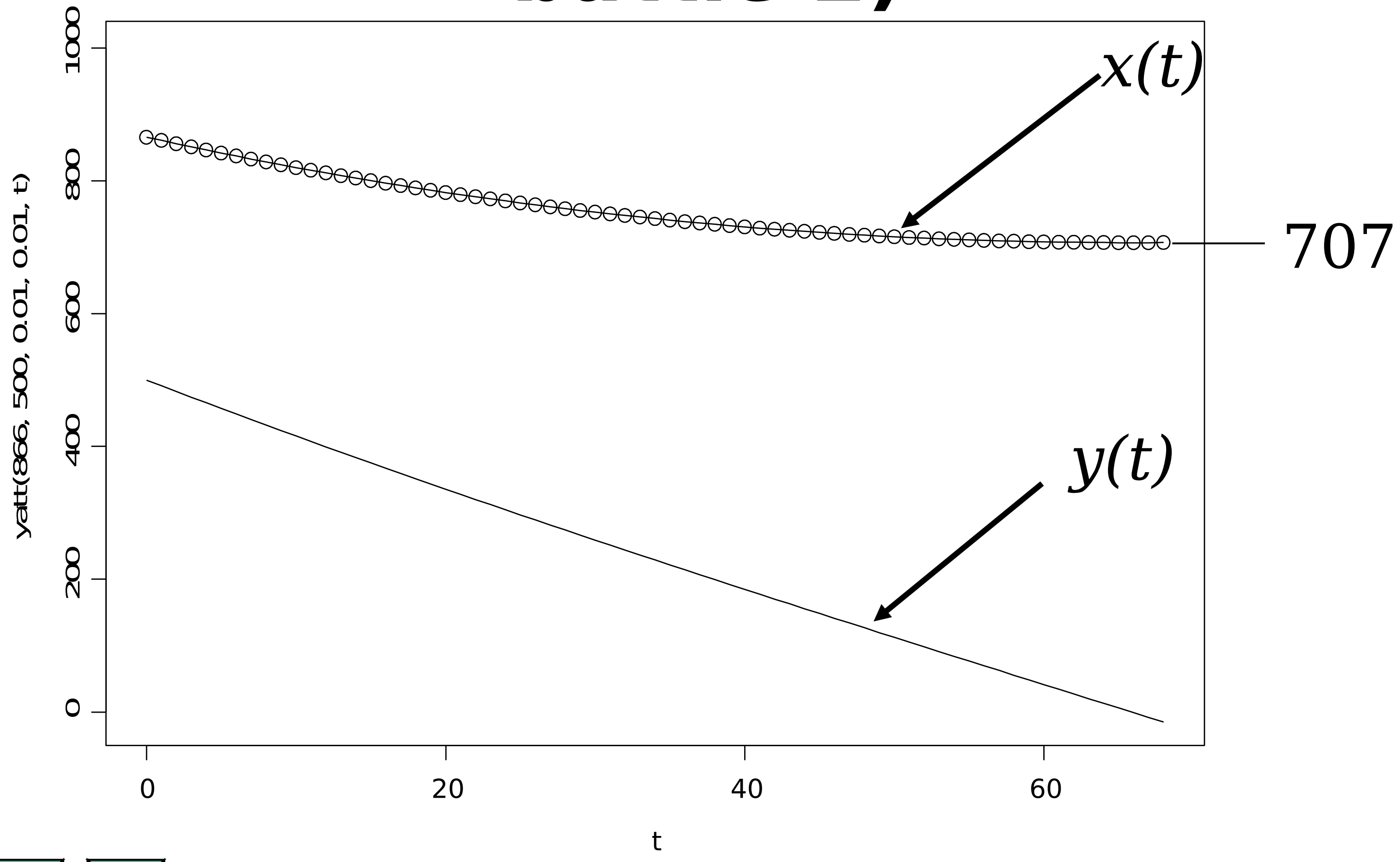
Example: Lanchester Square (divide and conquer, battle 1)

- $x_0 = 1000$, $y_0 = 500$, $a = .01$, $b = .01$



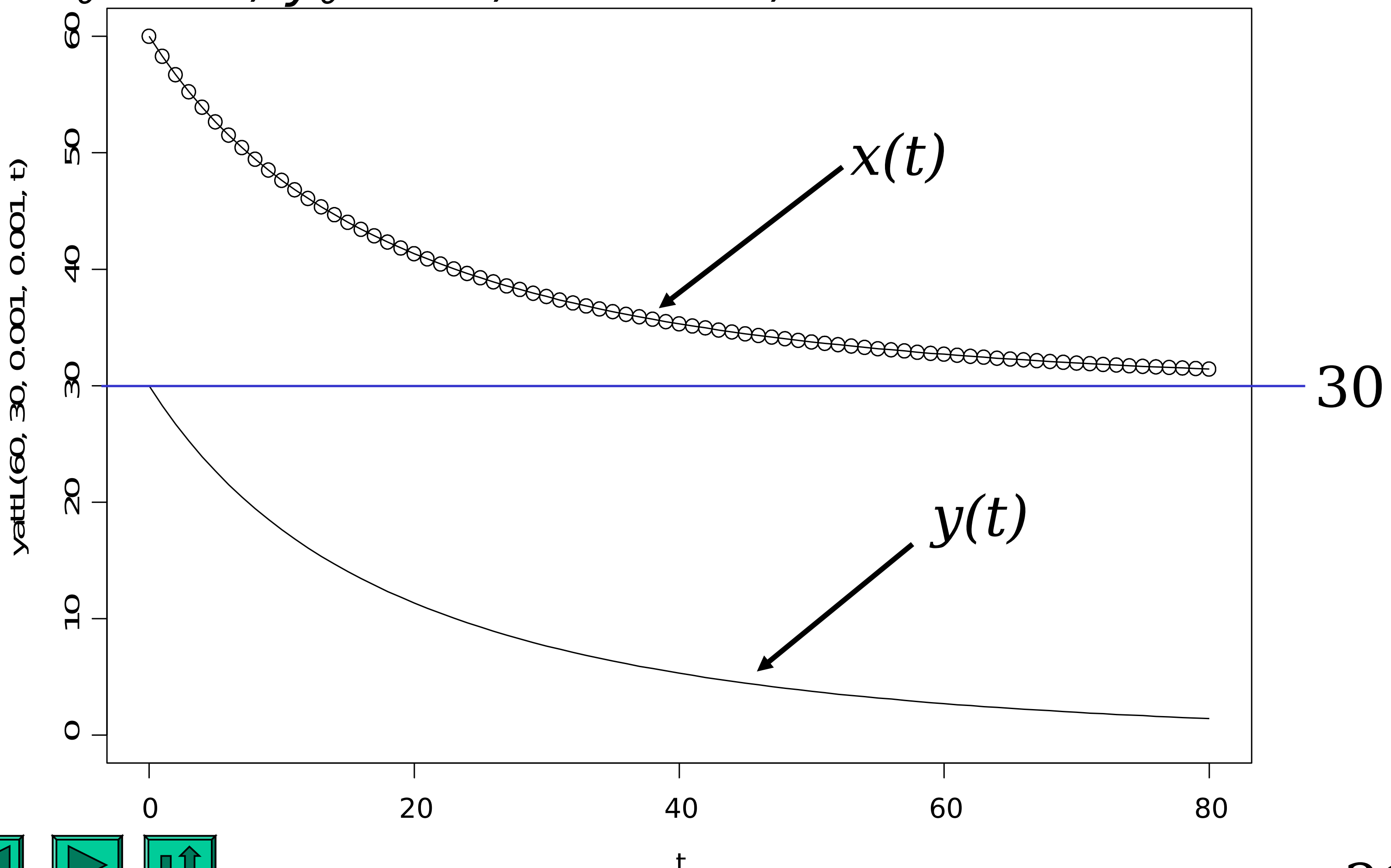
Example: Lanchester Square (divide and conquer, battle 2)

- x_0 = 866, y_0 = 500, a = .01, b = .01



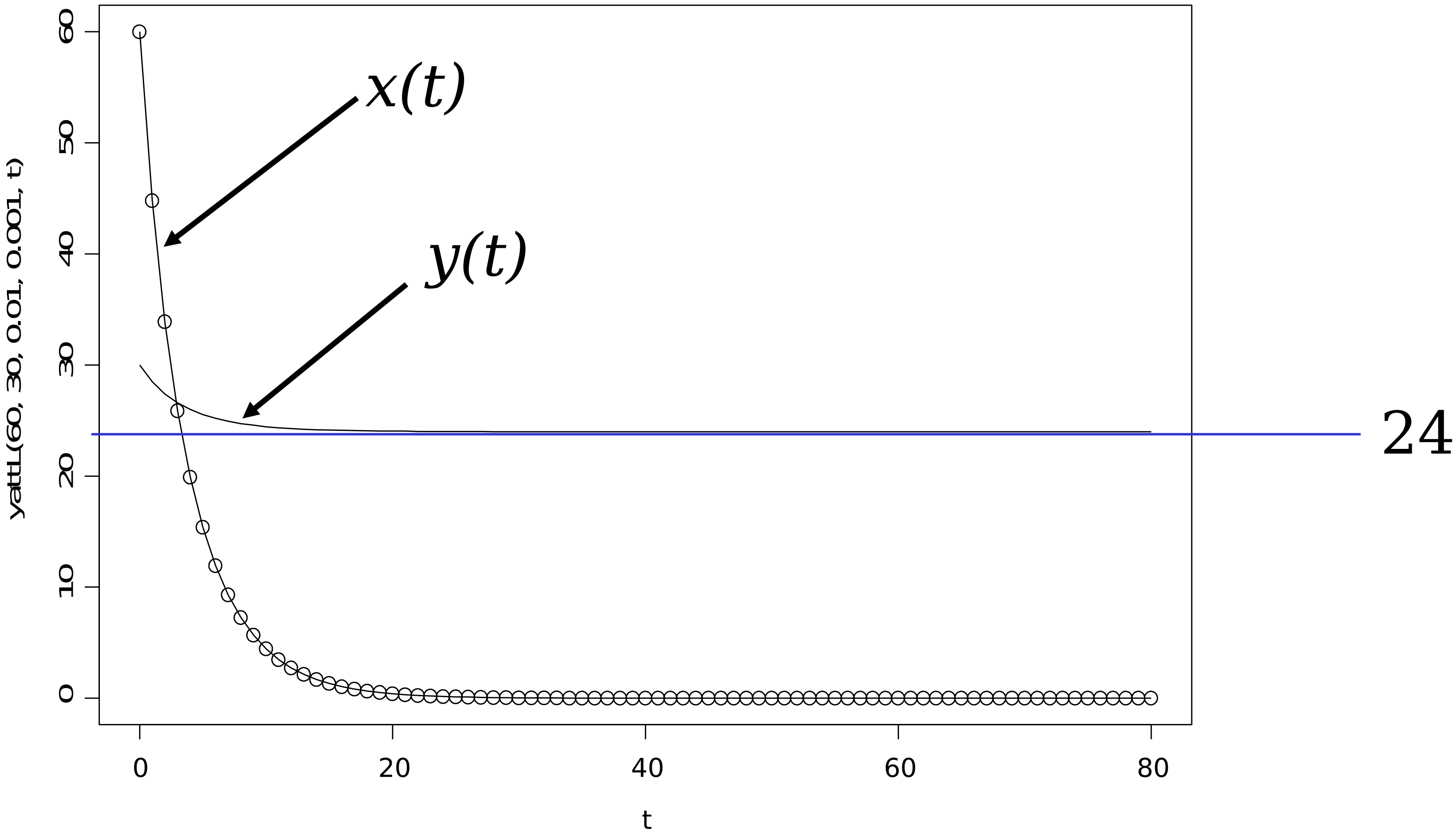
Example 1: Lanchester Linear

- x_0 = 60, y_0 = 30, a = .001, b = .001



Example 1: Lanchester Linear

- x_0 = 60, y_0 = 30, a = .01, b = .001



Some Limitations of Lanchester

- Homogeneous forces
- No explicit spatial effects
- No explicit force movement
- Constant attrition-rate coefficients
- No element of “luck” or randomness
- Do not explicitly include
  - Logistics
  - Command and Control
  - Terrain
  - Adaptation
  - Etc. etc. etc.

So what are they goo

Lanchester and Iwo Jima

126

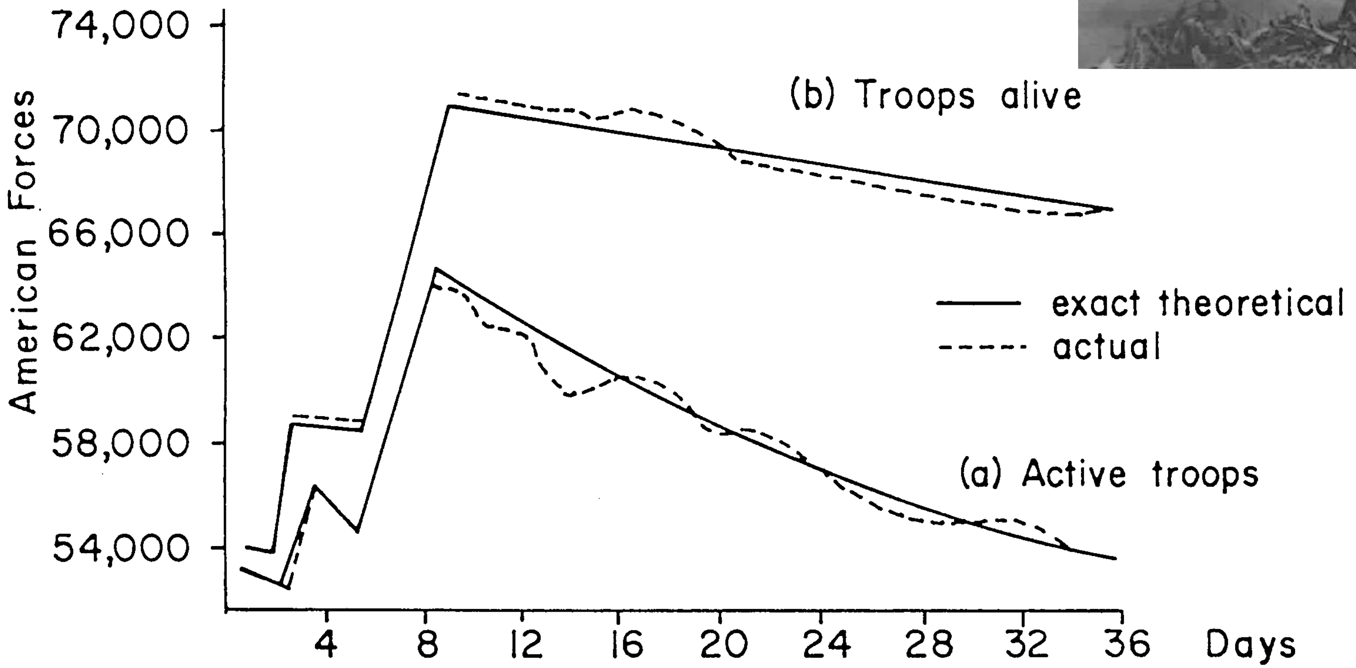


Figure 8.6. Comparison of Actual Troop Strength with Theoretical for the Battle of Iwo Jima (Adapted from Engel [3])

Recent work is not as supportive (stay tuned)…

Lanchester Equations and Battle Calculations

- We will now evaluate Lanchester equations to see who wins a battle, how long it takes, and how many survivors are left.

- First, recall Lanchester's two Models
  - Modern Combat ==> $\boxed{\frac{dx}{dt} = -ay \quad and \quad \frac{dy}{dt} = -bx}$
  - Ancient Combat ==> $\boxed{\frac{dx}{dt} = -axy \quad \text{and} \quad \frac{dy}{dt} = -bxy}$

Some Questions We Will Answer

Information from Lanchester's two simple analytical models

- Who will win the battle; or which force will be annihilated?
- What force ratio is required to guarantee victory?
- How many survivors will the winner have?
- How long will the battle last?
- How do the force levels change over time?
- How do changes in the parameters {e.g. initial force levels (x_o and y_o) or attrition coefficients (a and b)} affect the outcome of the battle?
- Is concentration of forces a good tactic?

Let’s Do “Modern Combat” First

- The state equation: $b\left(x_o^2 - x(t)^2\right) = a\left(y_o^2 - y(t)^2\right)$

- *X* wins a “fight-to-the finish” if, when *y(t) = 0*, *x(t) > 0*

- Rewrite the state equation: $x_f = \sqrt{x_o^2 - \frac{a}{b} y_o^2}$

  – where x_f = *X* level at the finish

  – where y_f = *Y* level at the finish

- Result: $x_f > 0$ <==> $\frac{x_0}{y_0} > \sqrt{\frac{a}{b}}$

"Victory Conditions" for a Fight-to-the-Finish

- *X* wins if: $\frac{x_0}{y_0} > \sqrt{\frac{a}{b}}$
- *Y* wins if: $\frac{x_0}{y_0} < \sqrt{\frac{a}{b}}$
- A draw if: $\frac{x_0}{y_0} = \sqrt{\frac{a}{b}}$

- In this set up, if your opponent outnumbers you by a factor of two, how much better individual effectiveness do you need to win? A factor of three?

Force Levels as a Function of Time

- Need to solve: $\boxed{\dfrac{dx}{dt} = -ay \quad and \quad \dfrac{dy}{dt} = -bx}$

-- with initial conditions $x(0) = x_0$ and $y(0) = y_0$

- Let's do it (one of the few we can solve)

- Solution: $\boxed{x(t) = \dfrac{1}{2}\left(\left(x_o - \sqrt{\dfrac{a}{b}}\,y_o\right)e^{\sqrt{ab}\,t} + \left(x_o + \sqrt{\dfrac{a}{b}}\,y_o\right)e^{-\sqrt{ab}\,t}\right)}$

$$\boxed{y(t) = \frac{1}{2}\left(\left(y_o - \sqrt{\frac{b}{a}}\,x_o\right)e^{\sqrt{ab}\,t} + \left(y_o + \sqrt{\frac{b}{a}}\,x_o\right)e^{-\sqrt{ab}\,t}\right)}$$

- Note the symmetry in the answer
- Note: "Victory Conditions" can be obtain from these

Which You Immediately Recognize as

$$\boxed{x(t) = x_o \cosh(\sqrt{ab}\ t) - y_o \sqrt{\frac{a}{b}} \sinh(\sqrt{ab}\ t)}$$

$$\boxed{y(t) = y_o \cosh(\sqrt{ab}\ t) - x_o \sqrt{\frac{b}{a}} \sinh(\sqrt{ab}\ t)}$$

- With cosh(x) = $\frac{e^x + e^{-x}}{2}$ and sinh(x) = $\frac{e^x - e^{-x}}{2}$
- Note: $\frac{x(t)}{x_o} = \cosh(\sqrt{ab}\ t) - \frac{y_o}{x_o} \sqrt{\frac{a}{b}} \sinh(\sqrt{ab}\ t)$
- Which depends on:
  - Initial Force ratio $\frac{y_0}{x_0}$
  - effectiveness ratio $\sqrt{\frac{a}{b}}$
  - The intensity of the battle $\sqrt{ab}$

How Long Will the Battle Last (in a Fight-to-the-Finish)?

- Suppose Y wins, then we need to solve for t such that $x(t) = 0$

I.e., $$0 = \frac{1}{2}\left(\left(x_o - \sqrt{\frac{a}{b}}\,y_o\right)e^{\sqrt{ab}\,t} + \left(x_o + \sqrt{\frac{a}{b}}\,y_o\right)e^{-\sqrt{ab}\,t}\right)$$

- Note: use $z = e^{\sqrt{ab}\,t}$, and solve quadratic in z…SBES

- End up with: $$t_{Ywins} = \boxed{\frac{1}{2\sqrt{ab}}\ln\left(\frac{\left(\sqrt{\frac{a}{b}}\,y_o + x_o\right)}{\left(\sqrt{\frac{a}{b}}\,y_o - x_o\right)}\right)}$$

- Note: If X wins, use same equation, except switch a and b and x and y.

Order in Which to Solve Lanchester Square Law Problems

- Find out who wins
- Find out how long the battle lasts
- Find the number of survivors on each side

Let’s do an Example (Square Law)

- Suppose X_0 = 2000, Y_0 = 1000, a = .002, and b = .001.

- Find out who wins. $\frac{x_0}{y_0} ? \sqrt{\frac{a}{b}}$
  - X wins

- Find out how long the battle lasts: $t_{Xwins} = \frac{1}{2\sqrt{ab}} \ln\left(\frac{\sqrt{\frac{b}{a}}\, x_o + y_o}{\sqrt{\frac{b}{a}}\, x_o - y_o} \right)$
  - t battle = 623.225

- Find X survivors = $x_f = \sqrt{x_o^2 - \frac{a}{b} y_o^2}$
  - 1414.2

Plot of the Battle

2000

X force levels

Y force levels

time

Plot of the Battle Past *Y's* Annihilation

2000

X force levels

Y force levels

time

Ancient Combat (in less detail)

- The state equation: $b(x_0 - x(t)) = a(y_0 - y(t))$
- X wins a "fight-to-the finish" if, when $y(t) = 0$, $x(t) > 0$
- Rewrite the state equation: $x_f = x_o - \frac{a}{b} y_o$
  - where x_f = X level at the finish
  - where y_f = Y level at the finish
- Result: $x_f > 0$ <==> $\frac{x_0}{y_0} > \frac{a}{b}$

“Victory Conditions” for a Fight-to-the-Finish

- *X* wins if: $\dfrac{x_0}{y_0} > \dfrac{a}{b}$
- *Y* wins if: $\dfrac{x_0}{y_0} < \dfrac{a}{b}$
- A draw if: $\dfrac{x_0}{y_0} = \dfrac{a}{b}$

- In this set up, ability and numbers are equal

Force Levels as a Function of Time

- Need to solve: $\boxed{\frac{dx}{dt} = -axy \quad and \quad \frac{dy}{dt} = -bxy}$

with initial conditions $x(0) = x_o$ and $y(0) = y_o$

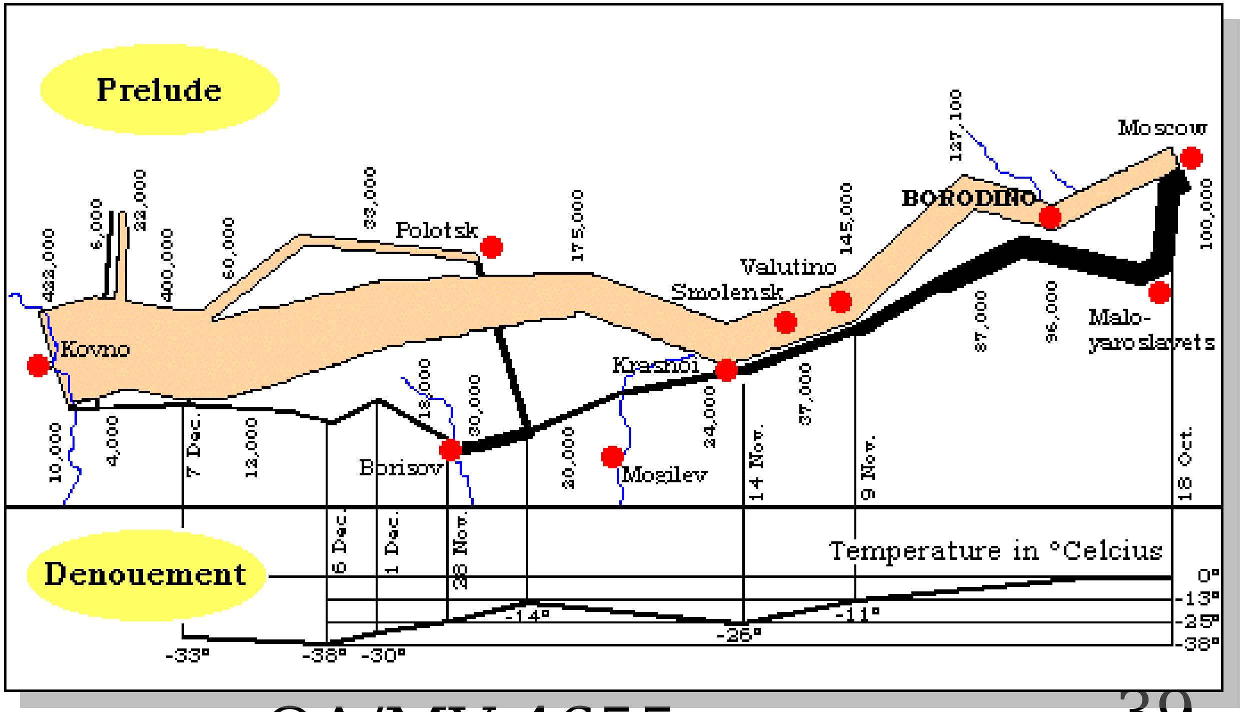
- Very hard, jump to answer

- Solution: $\boxed{x(t) = x_0 [\frac{bx_0 - ay_0}{bx_0 - ay_0 \times Exp\{-(bx_0 - ay_0)t\}}], \text{ if } bx_o \neq ay_o}$

$$\boxed{x(t) = [\frac{x_0}{1 + bx_0 t}], \text{ if } bx_o = ay_o}$$

Other Differential Equation Formulations

- Logarithmic law ==> $\frac{dx}{dt} = -ax \quad and \quad \frac{dy}{dt} = -by$
  - State equation => $b \times \ln\left(\frac{x_o}{x(t)}\right) = a \times \ln\left(\frac{y_o}{y(t)}\right)$



Other Differential Equation Formulations II

- Morse and Kimball Law

$$\boxed{\frac{dx}{dt} = P - ay - bx \quad \text{and} \quad \frac{dy}{dt} = Q - bx - ay}$$

- Could consider

$$\boxed{\frac{dx}{dt} = -ay - \alpha xy - \beta x + R(t)}$$

$$\frac{dm}{dt} = P - an - cm\,,$$

$$\frac{dn}{dt} = Q - bm - dn\,, \tag{12}$$

where, in general, a and b are larger than c or d. At first, we will consider the production rates, P and Q, to be constant. Differential equations[b] of this sort have been discussed in relation to the struggle between animal species[11,12] and in chemical kinetics. The unit of time can be the year. The quantity P will then be the number of equivalent armies (or the equivalent number of battleships) which the Red nation can train and equip in a year, and so on.

The solutions for equations (12) are:

$$m = A + Ee^{(\mu-\lambda)t} + \frac{\mu+\kappa}{b} Fe^{-(\mu+\lambda)t}\,;$$

$$n = B - \frac{\mu+\kappa}{a} Ee^{(\mu-\lambda)t} + Fe^{-(\mu+\lambda)t}\,;$$

$$\lambda = \tfrac{1}{2}(c+d)\,;\ \kappa = \tfrac{1}{2}(c-d)\,;\ \mu = \sqrt{\kappa^2 + ab}\,;$$

$$A = \frac{Qa - Pd}{ab - cd}\,;\ B = \frac{Pb - Qc}{ab - cd}\,;\ ab - cd = \mu^2 - \lambda^2\,;$$

$$E = \frac{ab}{2\mu(\mu+\kappa)}\left\{\left[m_0 + \frac{d+\mu+\kappa}{ab-cd}P\right] - \frac{\mu+\kappa}{b}\left[n_0 + \frac{c+\mu-\kappa}{ab-cd}Q\right]\right\}; \tag{13}$$

$$F = \frac{ab}{2\mu(\mu+\kappa)}\left\{\frac{\mu+\kappa}{a}\left[m_0 + \frac{d-\mu+\kappa}{ab-cd}P\right] + \left[n_0 + \frac{c-\mu-\kappa}{ab-cd}Q\right]\right\}.$$

Since ab is larger than cd, in general, we have μ greater than λ. Therefore, the exponential in the second term of the equations for m and n continually

Hembold Equations

$$\boxed{\frac{dx}{dt} = -a(x/y)^{1-w} y \quad \text{and} \quad \frac{dy}{dt} = -b(y/x)^{1-w} x}$$

- w is a measure of how well forces can concentrate t
  - » $w = 0$ ==> logarithmic law
  - » $w = 1$ ==> "square law"
  - » $w = 1/2$ ==> $-ax^{.5}y^{.5}$, get state equation linear lav
- Sometimes see $\boxed{\frac{dx}{dt} = -ax^P y^Q \quad and \quad \frac{dy}{dt} = -bx^Q y^P}$

Mixed Combat (Ambush)

$$\boxed{\frac{dx}{dt} = -axy \quad and \quad \frac{dy}{dt} = -bx}$$

- Think of one side (*X* = guerillas) engaging (ambushing) a conventional force (*Y*)
- State equation ==> $\boxed{2b(x_o - x(t)) = a(y_o^2 - y(t)^2)}$
- Victory conditions
  - The conventional force *Y* wins if $\dfrac{y_0}{x_0} > \sqrt{\dfrac{2b}{ax_0}}$

An example of mixed combat (as applied to Vietnam by Courtney Brown)

- Think of one side (X = guerillas) engaging (ambushing) a conventional force (Y)
- Some assumptions
  - x_0 = 100, b = rate of fire · P_{kill} (say = 1 · .1= .1)
  - y_0 = TBD, a = rate of fire · (A_L/A_T) (say = 1 · (2/100000) = .00002)
  - Need $$y_0 > x_0 \times \sqrt{\frac{2b}{ax_0}}$$
  - Need y_0 > 1000 or $$\frac{y_0}{x_0} > 10$$

Using a Simple Model to Support an Argument (1)

- Pictures from Brown's paper (Guerillas, Vietnam)

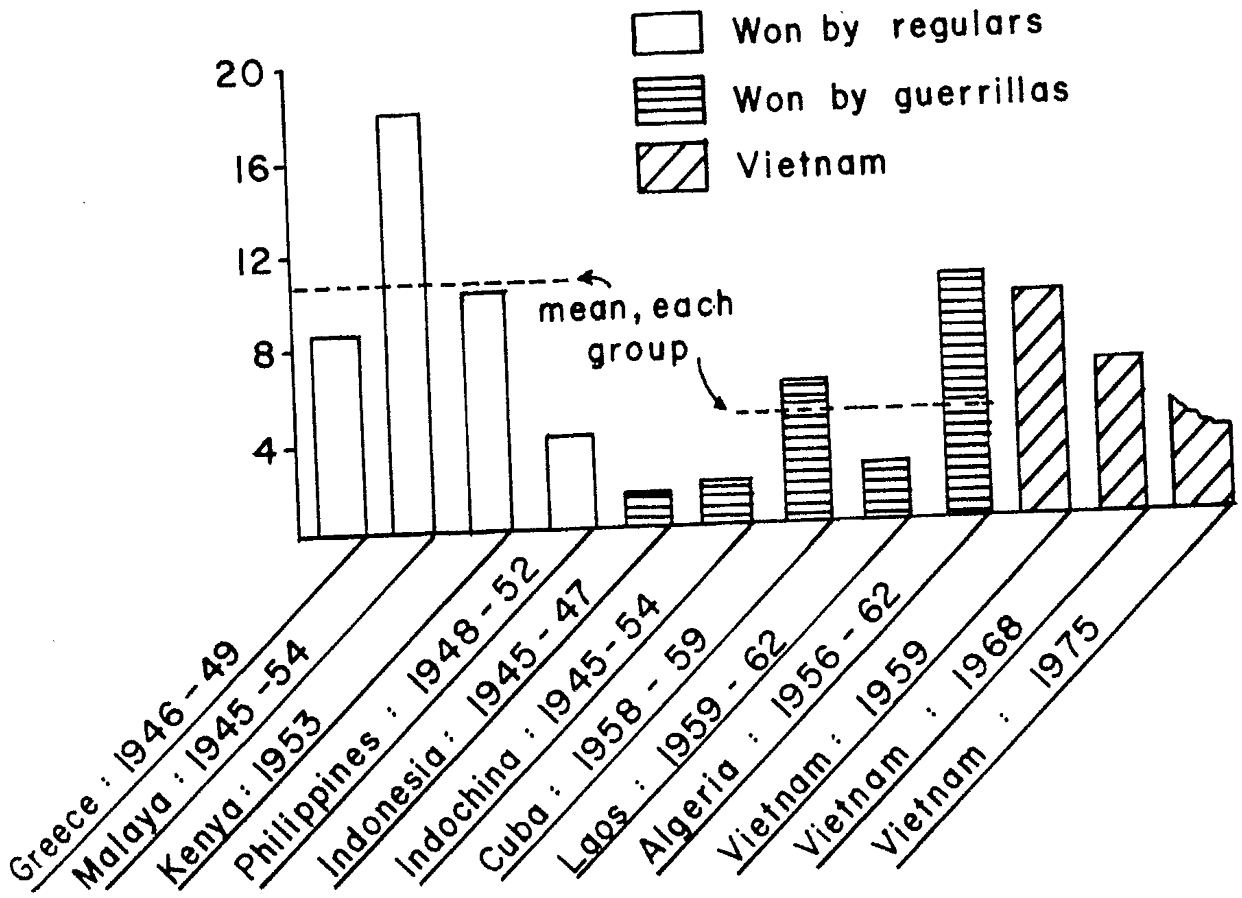


Figure 8.5. Mixed Guerrilla–Conventional Combats

Using a Simple Model to Support an Argument (2)

- Pictures from Brown's paper (Guerillas, Vietnam)

Forces in South Vietnam, Spring 1968

Conventional Forces		*Guerrilla Forces*	
American	510,000	North Vietnamese†	50,000
South Vietnamese: regulars	600,000	Viet Cong†	230,000
South Vietnamese: local defense†	500,000		
Other allies	70,000		
Total	1,680,000	Total	280,000

$$\text{Force ratio: } \frac{1{,}680{,}000}{280{,}000} = 6$$

† Approximate

If President Johnson had sent the 206,000 troops as requested, the force ratio would have increased to

$$\frac{1{,}886{,}000}{280{,}000} \sim 6.7,$$

still not enough to have improved the situation much for the conventional forces. Moreover, the guerrilla forces would only have had to increase to 314,000 in order to maintain a ratio of 6 : 1. It was analyses such as this, coupled with the disquiet of the American people about the whole affair, that led President Johnson to seek a political solution to the Vietnamese conflict. He rejected Westmoreland's request and initiated the Paris peace talks, which eventually led to the American disengagement of 1973. The final victory of the Viet Cong and North Vietnamese came in April 1975. (Let us again emphasize that we have ignored many significant factors such as the bombing of North Vietnam and the effect of the terrorist campaigns.)

Enriching Lanchester

- Reinforcements
- Breakpoints
- Range dependent attrition coefficients
- Etc.

Enriching Lanchester (reinforcements/break points)

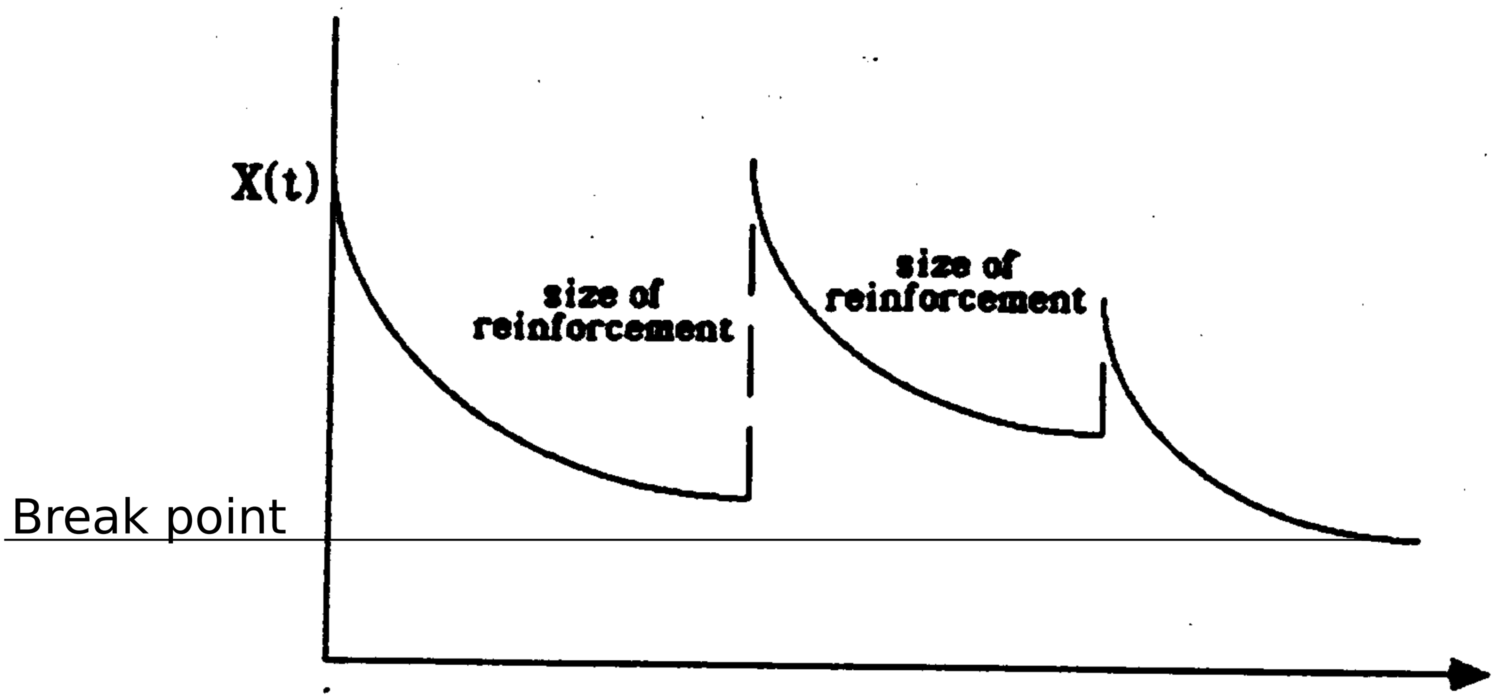


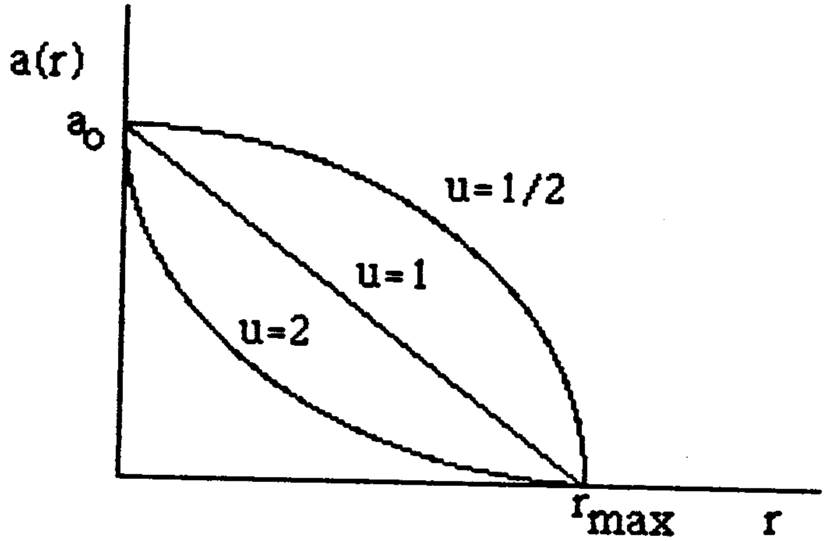
Figure 5.6.1 – Unit Reinforcement in Lanchester Models

Enriching Lanchester (range dependent attrition coefficients)

$$\boxed{\frac{dx}{dt} = -a(r)y \quad and \quad \frac{dy}{dt} = -b(r)x}$$

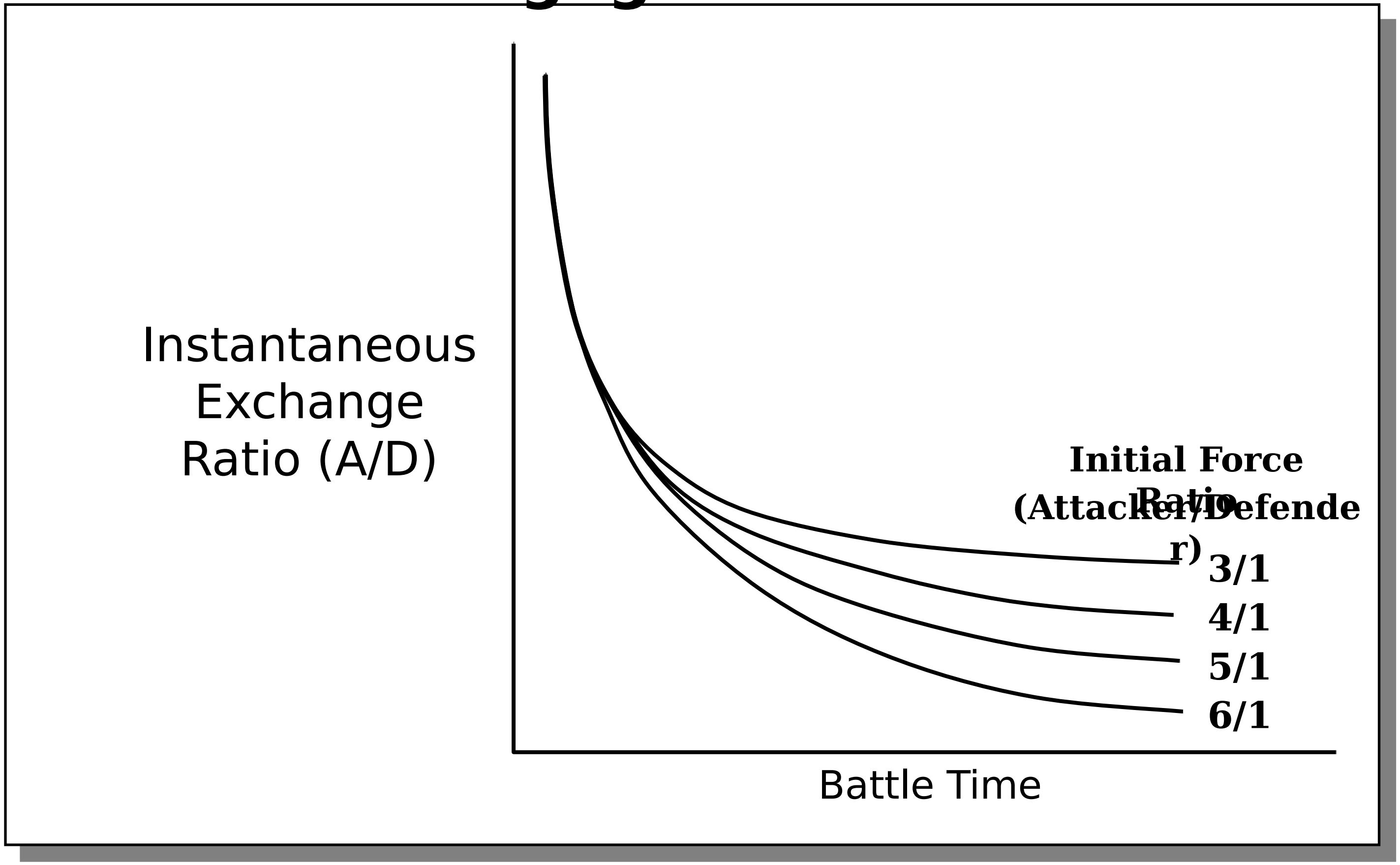
$$a(r) = a_o\left(1 - \frac{r}{r_{\max}}\right)^m \quad \text{for } 0 \leq r \leq r_{max}$$

$$a(r) = 0 \quad \text{for } r \geq r_{max}$$



(From Seth Bonder) Development 1965-1975:

Insights from Analyses of Small Unit Engagements



- Instantaneous Exchange Ratio as a Function of Battle Time

Solving Lanchester Equations Numerically

- Most Lanchester like formulations can be solved numerically with high accuracy.

Use:

$$-ay(t) = \frac{dx(t)}{dt} \equiv \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \cong \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

Thus: *--for the square law*

$$x(t+\Delta t) \cong x(t) - a \cdot y(t) \cdot \Delta t$$

Can do similar for *y(t+Δt)* and other formulations

Linear law =>

Logarithmic law => $x(t+\Delta t) \cong x(t) - a \cdot x(t) \cdot y(t) \cdot \Delta t$

$$x(t+\Delta t) \cong x(t) - a \cdot x(t) \cdot \Delta t$$

Example of Numerical Solution

- Square law with $x_0 = 100,\ y_0 = 200,\ a = .01,\ b = .02,\ \Delta t = .1$
- $x(.1) = x(0) - a \cdot y(0) \cdot \Delta t$

  $= 100 - .01 \cdot 200 \cdot .1$

  $= 99.8$
- $y(.1) = y(0) - b \cdot x(0) \cdot \Delta t$

  $= 200 - .02 \cdot 100 \cdot .$

  $= 199.8$
- Repeat each Δt…

  $x(.2) = x(.1) - a \cdot y(.1) \cdot \Delta t = 99.8 - .01 \cdot 199.8 \cdot .01 = 99.6002$
- Etc.
- Note: analytically $x(.2) = 99.6004$

Enriching Lanchester (adding decision thresholds)

- Infamous Dewar model. Square law is at the heart, with reinforce/withdraw decisions

	Blue	Red
Initial troop strength	$B_0 = 500$	R_0
Combat attrition calculation	$B_{n+1} = B_n - R_n/2048$	$R_{n+1} = R_n - B_n/512$
Reinforcement thresholds	$R_n/B_n \geq 4$ or $B_n < 0.8 \times B_0$	$R_n/B_n \leq 2.5$ or $R_n < 0.8 \times R_0$
Reinforcement block size	300	300
Allowable reinforcement blocks	5	5
Reinforcement delay (time steps)	70	70
Withdrawal thresholds	$R_n/B_n \geq 10$ or $B_n < 0.7 \times B_0$	$R_n/B_n \leq 1.5$ or $R_n < 0.7 \times R_0$

Heterogeneous Lanchester

- Typically each side has many weapon systems
  - $\underline{\boldsymbol{X}} = [x_1(t), x_2(t), \ldots, x_m(t)]$
  - $\underline{\boldsymbol{Y}} = [y_1(t), y_2(t), \ldots, y_n(t)]$
  - Where $x_i(t)$ = number of X survivors of weapon system i at time t
- Sometimes aggregate into force power (usually treat separately)
- A square law version

$d\underline{\boldsymbol{X}}/dt_{(m\times1)} = \underline{\boldsymbol{A}}_{(m\times n)} \cdot \underline{\boldsymbol{Y}}_{(n\times1)}$ and
$d\underline{\boldsymbol{Y}}/dt_{(n\times1)} = \underline{\boldsymbol{B}}_{(n\times m)} \cdot \underline{\boldsymbol{X}}_{(m\times1)}$

That is: $$\frac{dx_i}{dt} = \sum_{j=1}^{n} (\text{attrition of } x_i \text{ systems caused by } y_j \text{ systems})$$

- A total of $m+n$ differential equations
- Forget about closed form solutions…i.e., must be solve numerically!

Heterogeneous Lanchester (more details)

- A_{ij} = rate one *Y* system of type *j* kills *X* systems of type *i*
- $A_{ij} = \phi_{ij} \cdot a_{ij}$

where:

>> a_{ij} = rate one *Y* system of type *j* kills *X* systems of type *i* when all it fires at is *X* systems of type *i*

>> ϕ_{ij} = a tactical parameter (fire allocation) that is the fraction of *Y* systems of type *j* that are allocated to fire at *X* systems of type *i*

- Note: $\sum_i \phi_{ij} = 1$, for all *j*, if all are engaged.
- Note: get a_{ij} from labs, engineering data
  get ϕ_{ij} from tactical data, doctrine
- Solving numerically:

Square Law = $$x_i(t+\Delta t) = x_i(t) - \sum_{j=1}^{n} A_{ij} \cdot y_j(t) \cdot \Delta t$$

Linear law = $$x_i(t+\Delta t) = x_i(t) - \sum_{j=1}^{n} A_{ij} \cdot x_i(t) \cdot y_j(t) \cdot \Delta t$$

Example Solution

- General equation (square): $$\begin{pmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \end{pmatrix} = - \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

- Plug in numbers: $$\begin{pmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \end{pmatrix} = - \begin{bmatrix} .01 & .02 \\ .001 & .02 \end{bmatrix} \begin{pmatrix} 200 \\ 10 \end{pmatrix}$$

- Example, with $x_1(t) = 100$, $x_2(t) = 10$

  For $\Delta t = .1$, get $x_1(t+.1) = x_1(t) - .01 \cdot 200 \cdot .1 - .02 \cdot 10 \cdot .1$

  $\Rightarrow x_1(t+.1) = 100 - .2 - .02$

  $\Rightarrow x_1(t+.1) = 99.78$

Heterogeneous Lanchester (more details)

- For the heterogeneous model to function we have to make two assumptions about additivity and proportionality.
  - The first assumption, *additivity* says that there is no direct synergism. Simply stated the only way any antitank systems can contribute to the effectiveness of tank systems is by killing enemy tank systems. Consequently, their presence or absence in a force does not enhance the killing potential of a tank system. To model synergistic effects is a complex task however it is not a problem here as the additivity assumption has eliminated the possibility of such effects.
    - » Sam Parry calls theses models bean counters (as opposed to value)
  - The second assumption, *proportionality* says that *the loss rate* of x_i *caused by* y_j is *proportional to the number* of y_j that engage x_i. Think no diminishing returns, returns of scale, or threshold effects.

A More General Formulation

$$\frac{dx_i}{dt} = -F1(\vec{X}_{m\times 1}, \vec{Y}_{n\times 1}, \text{t}, \text{other factors}),\ \forall\ \text{i}$$

$$\frac{dy_j}{dt} = -F2(\vec{X}_{m\times 1}, \vec{Y}_{n\times 1}, \text{t}, \text{other factors}),\ \forall\ \text{j}$$

- These usually have stable numerical solutions and are easy to evaluate numerically

Some Brief Words on Estimating Attrition Coefficients

- Attrition coefficients (a, b, $A_{(mxn)}$, $B_{(nxm)}$) are critical and depend on lots of factors (type of battle, weather, terrain, doctrine, ability, morale, etc.) and change over time.

- Breadth of ways of getting
  - Theory or hypothesis
  - Historical (Dupuy)/training data
  - Clark's MLE estimates from detailed models
    - Or ATCAL (extrapolation from hi-res model)
  - Bonder's theoretical Markov Chain based

"Naïve" Estimates

- Aimed fire
  - a = (firing rate) · (prob. of a casualty per shot by one y)

    or $a = \nu_f \cdot P_{ssk}$ (can get data to estimate)

    so, $dx/dt = -\nu_f \cdot P_{ssk} \cdot y$

- Area fire
  - a = (firing rate) · (prob. of a casualty per shot per x by one y)

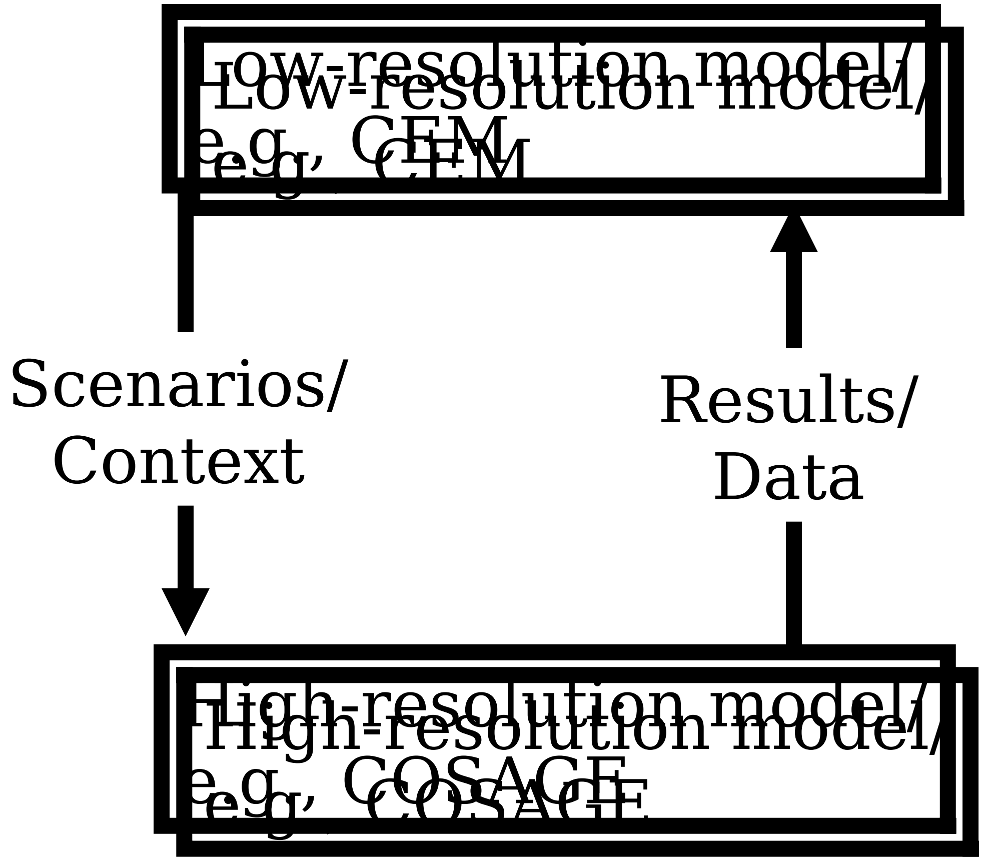
    or $a = \nu_f \cdot (A_L/A_T)$

    so, get $dx/dt = -\nu_f \cdot (A_L/A_T) \cdot x \cdot y$

Two Approaches in Practice: (1) COMAN

- Gordon Clark’s COMAN (Combat Analysis Model) [1969]
  - “Fitted parameter model”
  - Main idea: Use detailed hi-resolution model and maximum likelihood estimates (MLE) to fit low-resolution Lanchester model
  - Used by CAA (CEM and COSAGE)
- Strengths:
  - Coefficient estimates from low resolution models should capture synergies between weapon systems
- Weaknesses:
  - Estimates only as good as the hi-resolution model
  - Need to have pre-run enough scenarios so that the low-resolution model’s parameters are taken from appropriate battles. Lots of combinations of attack/defense forces, terrain, postures.

Picture of COMAN Idea



Two Approaches in Practice: (2) Bonder-Farrell

- Seth Bonder's model [1967]
  - "Self contained independent analytical"
  - Main idea: a = 1/E[Time to kill]. Use engineering level data to make calculations on the time to kill!
  - Used by VIC (likely in JWARS)
- Strengths:
  - Based on real understandable data
  - Standalone (i.e., does not rely on other models)
- Weaknesses:
  - Hard to capture synergies (built on a series of one-on-one calculations). Interactions between systems are not explicitly represented within this process.

From Seth Bonder: Origins 1960-65

Background: SB Introduction to Defense Research

- Worked in Systems Laboratory on Contract with US Army Armor School at
  Fort Knox Performing Modeling and Experimental Research on System *Performance Characteristics* and Their Interactions
  - Visual Detection — (Field Tests, Navy Labs)
  - Pinpoint Detection (Field Tests)
  - Firing Accuracy (BRL, Frankford)
  - Loading Times (Field Tests)
  - Firing Times (Field Tests)
  - Lethality (BRL)
  - Ballistic Protection (BRL)
  - Cross Country Speed (Theory, Engineering Tests)
  - Cruising Range
    - •
    - •
    - •
- Request for Support on Sheridan — What Performance Characteristics *Should* it Have?

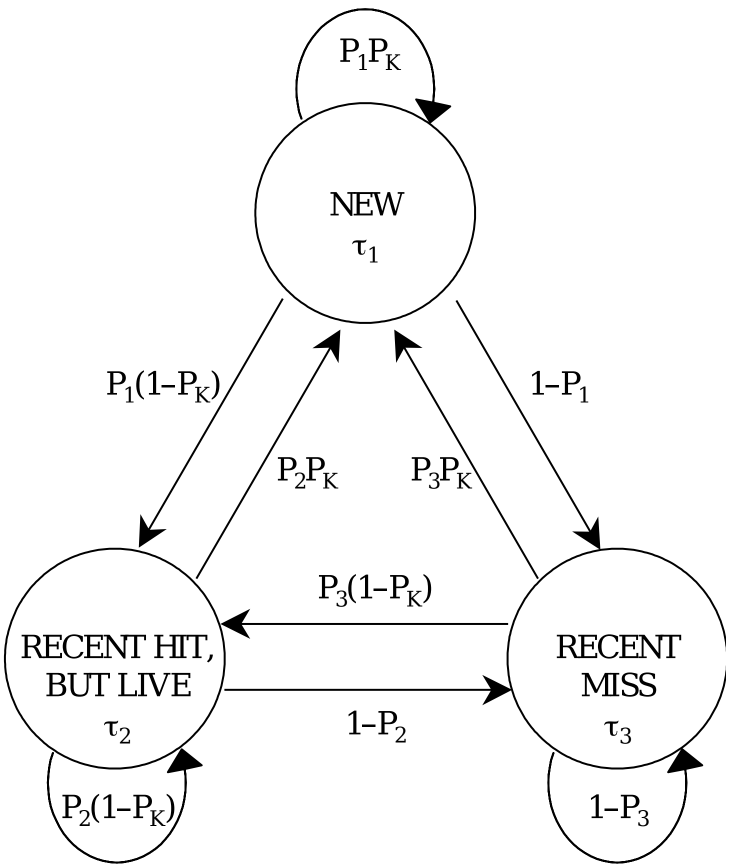
Simple Independent Repeated Shot Model

- Let T = time to kill
- $T = t_a + t_{k|a}$
  - t_a = time to acquire
  - $t_{k|a}$ = time to kill given acquire
- Assume that the firer shoots at a fixed rate of fire and with constant effectiveness until the target is killed (with each shot independent of others).
- Let P_s = probability of a kill for each shot = $P_h \times P_{k|h}$
- The number of shots (N) until the target is killed ~ geometric(P_s)
- $E(N) = 1/P_s$ (prove it!)
- Let t_s = time for each shot
- Then, $E[t_{k|a}] = t_s / P_s$
- So, $E[T] = t_a + t_s / P_s$

Let’s do an example…

- Suppose:
  - The mean time to acquire the target (t_a) = 120 seconds
  - The time to fire each round (t_s) = 10 seconds
  - The probability of a kill for each shot (P_s) = .4
- Then,
  - $E[T] = t_a + t_s/P_s = 120 + 10/.4$
  - So, $E[T] = 145$ seconds
  - Therefore, $a = 1/E[T] = 1/145 = .0069$

Simple Markov Chain underlying the Bonder-Farrell Equations



- P_1 = probability of a first round hit.
- P_2 = probability of a hit on the current shot given a hit on previous shot.
- P_3 = probability of a hit on current shot given a miss on previous shot.
- P_K = probability of a kill given a hit.

- t_a = time to acquire a new target.
- t_1 = time to fire the first round after target acquisition.
- t_h = time to fire a round following a hit.
- t_m = time to fire a round following a miss.
- t_f = projectile flight time to target.

State Occupation times

$\tau_1 = t_a + t_1 + t_f$

$\tau_2 = t_h + t_f$

$\tau_3 = t_m + t_f$

Solve for the mean time to return to state "NEW" by Markov Chain renewal theory to get E(Time to kill)…somewhat messy formula

Solving For the Time to Return to State 1—(i.e., looking for a new target)

- Let x_i be the mean time to get from state *i* to state 1
- Thus, x_1 is the mean time to *return* to state 1 from state 1; this is the mean time between target kills.
- Three equations and three unknowns

$$x_1 = \tau_1 + P_1(1 - P_K)\, x_2 + (1 - P_1)\, x_3$$

$$x_2 = \tau_2 + P_2(1 - P_K)\, x_2 + (1 - P_2)\, x_3$$

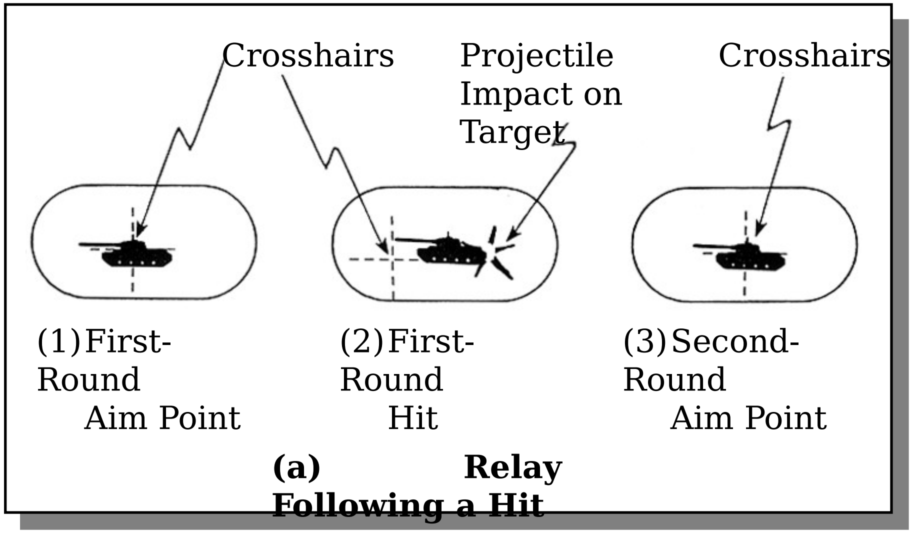
$$x_3 = \tau_3 + P_3(1 - P_K)\, x_2 + (1 - P_3)\, x_3$$

- SBES (not trivial) yields:

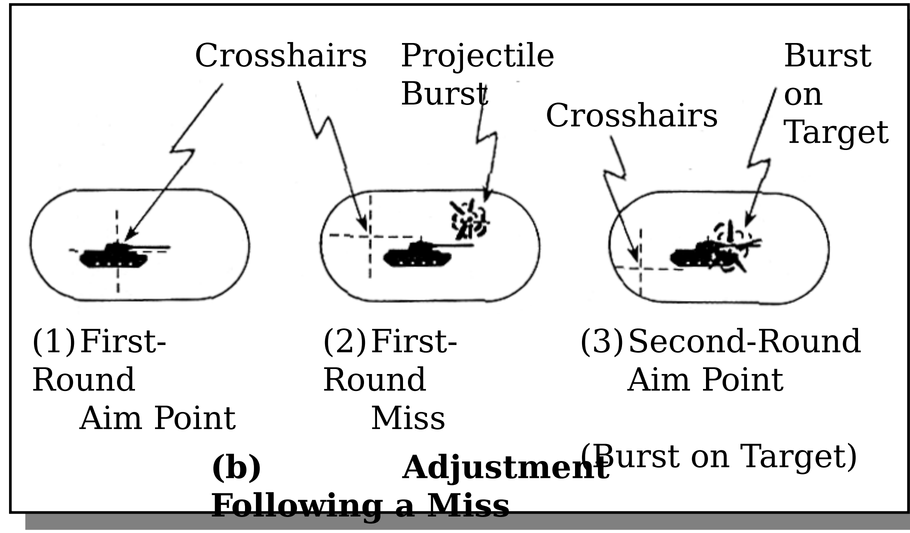
$$\boxed{x_1 = \tau_1 - \tau_2 + \frac{\tau_2}{P_K} + \frac{\tau_3}{P_3}\left(\frac{1 - P_2}{P_K} + P_2 - P_1\right)}$$

From Seth Bonder: Origins 1960-65

Attrition Rate for Tank Main Gun



(a) Relay Following a Hit



(b) Adjustment Following a Miss

Firing Doctrine for Tank Main Gun

- Developed pdf for Random Variable “Time to Destroy a Target”
- Then $E[T] = t_1 - t_2 + \frac{t_2}{P_K} + \frac{t_3}{P_3}\left(\frac{1 - P_2}{P_K} + P_2 - P_1\right)$
- And, $a = 1/E[T]$

Additional Resources

See the references below for additional material on Lanchester equations

- **Operations Research Department, *Aggregated Combat Modeling: Chapter 5*, Naval Postgraduate School**

- Other
  - Lanchester, F. W., *Aircraft in Warfare : The Dawn of the Fourth Arm* (reprints of 1916 book exist)
  - Taylor, J., *Lanchester Models of Warfare, Volume I*, Military Applications Section, Operations Research Society of America, 1983.
  - Taylor, J., *Lanchester Models of Warfare, Volume II*, Military Applications Section, Operations Research Society of America, 1983.

Glossary

- **Homogeneous forces:** All of the elements in the force are of the same type (e.g., manpower or tanks). Heterogeneous forces are often modeled in homogeneous equations by representing each unit in terms of combat power. For example, in a given scenario, an M1A1 tank might be worth 40 units of combat power, a M2A3 10 units of combat power, an infantry soldier, 1 unit of combat power, etc.

Deriving the Square Law State Equation: Use "Separation of Variables"

Given: $\dfrac{dx(t)}{dt} = -ay(t)$ and $\dfrac{dy(t)}{dt} = -bx(t)$

Divide equations: $\dfrac{dx(t)}{dy(t)} = \dfrac{ay(t)}{bx(t)}$

Separate variables: $bx(t)dx(t) = ay(t)dy(t)$

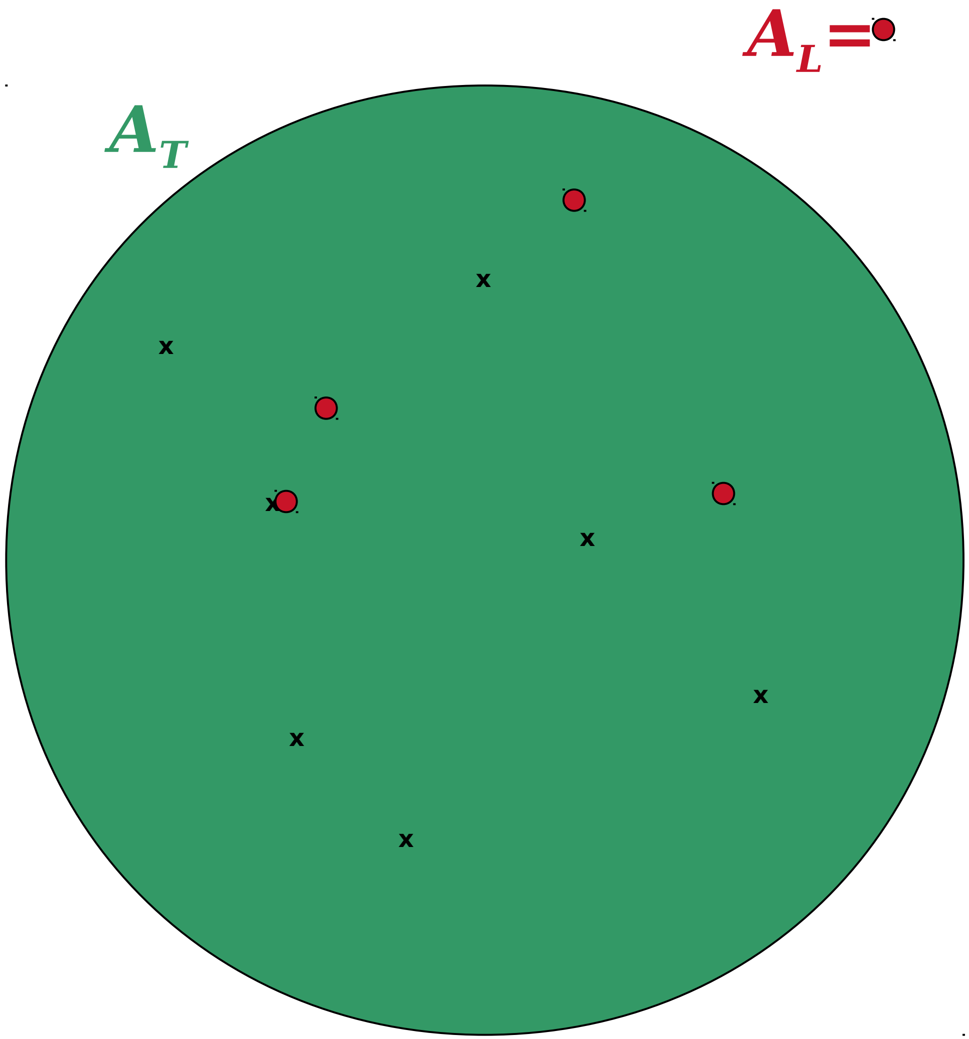
Return

Integrate: $\int bx\,dx = \int ay\,dy$

Yields: $\dfrac{bx(t)^2}{2} = \dfrac{ay(t)^2}{2} + c$, use $x(0) = x_0$ and $y(0) = y_0$ to find c

Some algebra [SBES] yields: $b(x_o^2 - x(t)^2) = a(y_o^2 - y(t)^2)$

“Area Fire” Interpretation



(1) P_{kill} of a given *X* per shot = A_L/A_T

(2) E[*X* killed per shot] = $(A_L/A_T) \cdot x$

(3) Let α = rate of shots/per *y*.

(4) Rate of shots = $\alpha \cdot y$

(5) Hence, expected rate *X* killed = $\alpha \cdot (P_{kill}$ a given $X = A_L/A_T) \cdot x \cdot y$

(6) Can interpret $a = \alpha \cdot$

Return

(7) $= a \cdot x \cdot y$

Lanchester Square Law Battle Trace Solution

Given: $\frac{dx(t)}{dt} = -ay(t)$ and $\frac{dy(t)}{dt} = -bx(t)$; and $x(0) = x_0$, $y(0) = y_0$

Note: $\frac{dx}{dt} = -ay$, so $\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(-ay\right) = -a\frac{dy}{dt}$

But: $\frac{dy}{dt} = -bx$, so $\frac{d^2x}{dt^2} = abx$ This is a second order Diff. Ec

Typical solution: Guess at the answer and show it work

Try: $x(t) = e^{rt}$, then $\frac{dx}{dt} = r\,e^{rt}$ and $\frac{d^2x}{dt^2} = r^2 e^{rt}$

Lanchester Square Law Battle Trace Solution continued

Therefore: $r^2 e^{rt} = ab\,x(t)$, note: $x(t)$ is e^{rt}

So: $r^2 = ab$, thus $r = \pm\sqrt{ab}$ works

Good general form guess is thus: $x(t) = Ae^{\sqrt{ab}\,t} + Be^{-\sqrt{ab}\,t}$

Need to determine constants A and B

Use initial conditions:

(1) $x(0) = x_0$

(2) $\frac{dx}{dt} = A\sqrt{ab}e^{\sqrt{ab}\,t} - B\sqrt{ab}e^{-\sqrt{ab}\,t}$, and $\frac{dx(0)}{dt} = -ay_0$

Yielding: $x_0 = A + B$ and $-ay_0 = A\sqrt{ab} - B\sqrt{ab}$

Lanchester Square Law Battle Trace Solution continued II

This leaves two equations in two unknowns…
Which can easily be solved by the energetic student (SBES) to yield

$$A = \frac{1}{2}\left(x_o - \sqrt{\frac{a}{b}}\, y_o\right), \text{ and } B = \frac{1}{2}\left(x_o + \sqrt{\frac{a}{b}}\, y_o\right)$$

Return

Future improvements

- More on ATCAL
  - Example on Bonder, ATCAL?
- More on Ambush transition (Shaffer)
- Breakpoint results (Most battles are not fights-to-the-finish)

Square Law - Breakpoint Calculations

- Victory Conditions
  - X wins IFF

$$\frac{x_0}{y_0} > \sqrt{\frac{a}{b}\left(\frac{1-\frac{y_{BP}^2}{y_0^2}}{1-\frac{x_{BP}^2}{x_0^2}}\right)}$$

$$t_{Xwins} = \begin{cases} \frac{1}{\sqrt{ab}} \ln \frac{x_0}{x_{BP}} & \text{if } \frac{x_0}{y_0} = \sqrt{\frac{a}{b}} \\ \frac{1}{\sqrt{ab}} \ln \left(\frac{y_{BP} - \sqrt{y_{BP}^2 - y_0^2 + \frac{b}{a} x_0^2}}{y_0 - \sqrt{\frac{b}{a}} x_0} \right) & \text{otherwise} \end{cases}$$

Mean of Geometric RV

- Let N ~ Geometric(p)
- $f_N(N=n) = (1-p)^{(n-1)} \cdot p$
- Let $q = 1-p$
- $E(N) = 1 \cdot p + 2 \cdot q \cdot p + 3 \cdot q^2 \cdot p + 4 \cdot q^3 \cdot p + \ldots$
- $E(N) = p \cdot (1 + 2 \cdot q + 3 \cdot q^2 + 4 \cdot q^3 + \ldots)$
  - Multiply both sides by $(1-q)$
- $E(N) \cdot (1-q) = p \cdot (1 - q + 2 \cdot q - 2 \cdot q^2 + 3 \cdot q^2 - 3q^3 + 4 \cdot q^3 + \ldots)$
- $E(N) \cdot (1-q) = p \cdot (1 + q + q^2 + q^3 + \ldots)$
- $E(N) \cdot (1-q) = p \cdot (1/(1-q))$
- $E(N) = p \cdot (1/(1-q)^2) = p \cdot (1/(p)^2) = 1/p$

Return

- Add numeric examples where possible (Bonder-Ferrell, simple + with formula, ATCAL like with KV scoreboard—section?) Note other BF formulations

- Link more glossary stuff?